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CRACK EXTENSION AND ENERGY RELEASE RATES IN FINITELY DEFORMED SHEETS REINFORCED WITH INEXTENSIBLE FIBRES

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Abstract—Plane stress of a torn sheet of finite extent is considered. The tear or crack, which is unstressed, runs from the boundary and is of fairly arbitrary shape. The sheet is reinforced by a continuous network of two families of inextensible fibres; this models a coated fabric or any other highly anisotropic material with two "strong" directions. The sheet is finitely deformed under in-plane dead loading of its boundary. For much of the paper the stiffness of the sheet, other than that contributed by the fibres, is assumed negligible compared with the applied loads, thus highlighting the effects of the strong anisotropy; otherwise, the material response is taken to be non-linearly elastic. Expressions for the stresses at the crack-tip are obtained in terms of the boundary loading. Equivalent expressions are also obtained for the energy release rate when the tear advances in an arbitrary direction (not necessarily parallel to the previous direction of the crack). Except in the limit of infinitesimal elastic deformation, there is no simple relation between the stress intensities (as measured by the fibre forces) at the crack tip and the energy release rate. Three possible fracture criteria—analogous to those based on maximum stress intensity or energy release rate in linear elasticity—are discussed, and their implications are illustrated by analysing the example of a torn rectangular sheet under uniform biaxial tensile loading.

1. INTRODUCTION

The principal aims of this paper are to evaluate the stress effects of cracks in strongly anisotropic sheets undergoing finite deformation, to use the results to formulate simple fracture criteria analogous to those applied to isotropic bodies and to assess the relative merits of such generalisations. The anisotropic material considered is characterised by having two "strong" directions, for each of which the extensional modulus is much greater than the shear moduli associated with that direction. Such anisotropy can occur naturally, but is most frequently found in man-made fibre-reinforced composites in which a relatively weak matrix is reinforced by families of very strong fibres. The mechanical response of such a material is usually not just anisotropic but *highly* anisotropic, so that isotropic theory would not provide even a rough approximation to its behaviour under most loading conditions. Hence the validity of fracture criteria proposed on the basis of experience of fracture of isotropic or weakly anisotropic materials is particularly suspect, and it is important to determine how the strong anisotropy is reflected in the fracture properties of the material.

Conventional stress analysis for anisotropic materials (as described [1–4] for linear elasticity and [5, 6] for plasticity) is considerably more complex than in the corresponding isotropic theory, especially as the anisotropy becomes more pronounced. Various crack problems involving infinite or semi-infinite bodies of particular anisotropic elastic materials have been examined by a number of authors[7–12] and some analytical solutions have been determined. However, in general the equations require numerical solution, and for bodies of *finite* extent this is almost invariably the case. Even then there are few such solutions in the literature, and the author knows of none where large deformations are involved.

In order to obtain analytical solutions which would demonstrate qualitatively the different behaviour produced by strong anisotropy, it has been found fruitful in many cases to model such materials as "ideal fibre-reinforced materials". This model treats the "strong" directions as if they were *inextensible* cords which are convected with the deformations. The theory of incompressible elastic materials reinforced by inextensible cords was first developed by Adkins and Rivlin[13] (further references are given in [14]), and their basic ideas of continuously distributed inextensible fibres have now been incorporated and developed extensively in theories which include elastic, plastic or viscoelastic behaviours (comprehensive descriptions have been given by Spencer[15, 18]).

Such an idealisation highlights the difference in response of a strongly anisotropic body as compared with that of isotropic bodies. There are two examples which are particularly relevant to fracture, and both have been substantiated for real highly anisotropic elastic materials by reference to exact and asymptotic solutions in linear anisotropic elasticity theory[19–23]. The first is the occurrence of ***stress-channelling***, in which the stressing at any given point in a body strongly affects the stress conditions found at large distances away, along any fibre (i.e. strong direction) passing through that point. This feature implies that end effects are *not* now localised to one or two typical diameters' distance from the boundary, as in isotropic elasticity, but can possibly extend through the entire body. The second is the ideal theory's remarkable prediction[24] of *surfaces* of singular stress occurring, as distinct from the conventional isolated points of singular stress predicted in isotropic elasticity; in real highly anisotropic materials these surfaces correspond to thin layers of high direct stress across which the shear stress varies rapidly. The implications of such strong anisotropy for fracture properties and crack propagation in plates undergoing infinitesimal elastic deformations have already been studied by a number of authors. Of these, England and Rogers[25] and Thomas *et al.*[26] considered the effect of a crack in a plate of finite size reinforced by one family or two orthogonal families of fibres and they suggested failure criteria based on the fibre forces at the crack tips. Sanchez and Pipkin[27] obtained the energy release rate for straight cracks propagating normal to a fibre-family, and their results have been generalised by Pipkin and Rogers[28] to include arbitrary crack directions. The latter authors also compared the crack trajectories (not necessarily straight) predicted by both the critical fibre force and energy release rate criteria for fracture.

Relevant problems involving finite deformations are the inflation of a line crack in a plate reinforced by a single family of fibres[29] and the in-plane deformation of pure inextensible networks containing a hole[30].

In this paper we analyse the particular context of a sheet of finite extent with reinforcement by a continuous network of two families of inextensible fibres. The sheet is finitely deformed under the action of in-plane dead loading of its boundary (refer Fig. 1). An unstressed tear runs from this boundary, and we are interested in the conditions which determine whether or not the tear extends and, if it does, in which direction. For much of the paper the stiffness of the sheet, other than that contributed by the fibres, is assumed to be negligible compared with the applied loads; otherwise the response is taken to be elastic, with a strain energy function suggested by Pipkin[31].

In the next section we introduce the relevant kinematics and stress function. The notation adopted is that used by Pipkin[31] whose results, together with those of Rivlin[32] and Adkins[33], are quoted in this section. In Section 3 we determine the magnitudes of the stresses in the fibres passing through the crack-tip in an unstiffened sheet, and the energy release rate during crack extension is obtained in Section 4. The generalisation of these results to inextensible networks with elastic stiffening is given in Section 5 and is related to the equivalent linearised theory in the next section. In Section 7 we discuss the implications of the analysis in providing fracture criteria. The analysis is applied in Section 8 to the particular example of biaxial tensile loading of a torn, initially rectangular sheet. The final section is a discussion of the results described in the paper and in particular considers whether a critical energy release rate can provide an appropriate criterion for fracture of such highly anisotropic materials.

2. KINEMATICS AND STRESS

We consider a plane sheet with two families of inextensible reinforcing fibres that initially lie along the X- and Y-directions of a system of Cartesian axes whose origin O is located at the tip of the tear (refer Fig. 1). In the reference configuration the sheet occupies the region $\mathcal{R}_0$ with boundary Γ_0. The sheet is treated as a two-dimensional continuum, so that every line $X = \text{constant}$ or $Y = \text{constant}$ is a line of inextensible material. We assume no slipping occurs between fibres of the two families so that each point (X, Y) in the sheet has the same two fibres passing through it at each stage of the deformation; the directions of these two fibres are described by the two unit vectors $\mathbf{a}$ and $\mathbf{b}$.

In the plane deformations considered in this paper, the particle initially at $\mathbf{X} \equiv (X, Y)$ moves

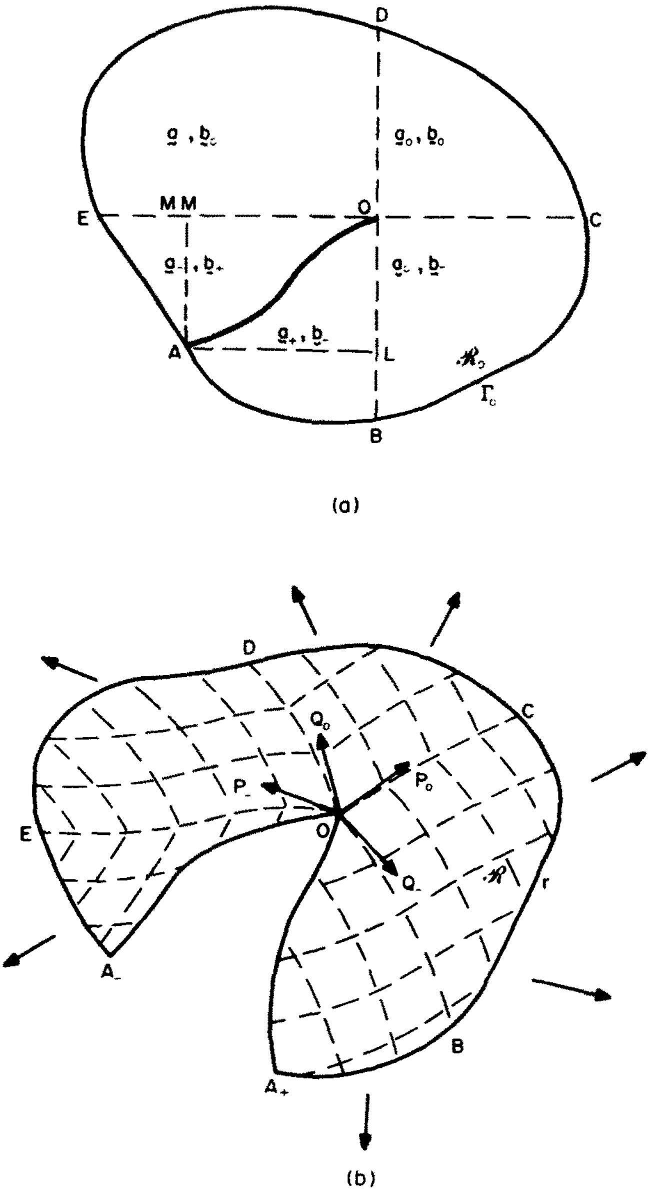

Fig. 1. (a) Undeformed configuration of the torn sheet. (b) Typical deformed configuration showing typical $\mathbf{a}$- and $\mathbf{b}$-fibres.

to $\mathbf{x}(X, Y)$, with

$$\mathbf{a} = \partial \mathbf{x}/\partial \mathbf{x}, \quad \mathbf{b} = \partial X/\partial Y, \tag{2.1}$$

so that $\mathbf{x}$ may be obtained from $\mathbf{a}$ and $\mathbf{b}$ by integrating

$$\mathrm{d}\mathbf{x} = \mathbf{a}\,\mathrm{d}X + \mathbf{b}\,\mathrm{d}Y. \tag{2.2}$$

Rivlin[32] showed that fibre-inextensibility implies that the fibre families must consist of congruent curves

$$\mathbf{a} = \mathbf{a}(X), \quad \mathbf{b} = \mathbf{b}(Y), \tag{2.3}$$

whenever $\mathbf{a}$ and $\mathbf{b}$ are not parallel and provided the fibres are unbroken. Hence the $\mathbf{b}$-direction, for example, is constant across the unbroken fibre $Y = \text{constant} > 0$, but across $Y = \text{constant} < O$

the **b**-fibres can (and usually do) take different constant directions depending on which side of the tear they are. In order to distinguish between the fibre-directions in the different regions of the sheet, we use the notation shown in Fig. 1; so in the segment OCD the fibre directions are $\mathbf{a}_0(X)$ and $\mathbf{b}_0(Y)$, whereas in ODE the directions are $\mathbf{a}_-(X)$ and $\mathbf{b}_0(Y)$.

For simplicity we exclude problems in which folds [31] occur; hence the current area J per unit initial area must be positive at every point, with

$$J = \mathbf{a} \times \mathbf{b} \cdot \mathbf{k}. \tag{2.4}$$

Here $\mathbf{k}$ is the unit vector in the Z-direction, perpendicular to the sheet.

For later use we introduce the vectors $\mathbf{A}(Y)$ and $\mathbf{B}(X)$ defined by

$$\mathbf{A} = \mathbf{b} \times \mathbf{k}, \quad \mathbf{B} = \mathbf{k} \times \mathbf{a}, \tag{2.5}$$

and note that

$$\mathbf{A} \cdot \mathbf{b} = O, \quad \mathbf{B} \cdot \mathbf{a} = O. \tag{2.6}$$

Suffices will refer to the relevant **a** or **b** fields, so that $\mathbf{A}_0 = \mathbf{b}_0 \times \mathbf{k}$ and $\mathbf{B}_- = \mathbf{k} \times \mathbf{a}_-$, for example.

The equilibrium state of stress in the deformed sheet is conveniently described [31] in terms of a stress potential $\mathbf{F}(X, Y)$ introduced by Rivlin [32]. This is defined such that the force exerted across a directed arc by the material originally to its right is the difference between the values of $\mathbf{F}$ at the two end points of the arc. Then $\partial\mathbf{F}/\partial Y$ and $\partial\mathbf{F}/\partial X$ are stress vectors with

$$\partial\mathbf{F}/\partial Y = T_a\mathbf{a} + S\mathbf{b}, \quad \partial\mathbf{F}/\partial X = -T_b\mathbf{b} - S\mathbf{a}. \tag{2.7}$$

Here the components T_a and T_b represent tensions in the fibre directions and are reactions to the kinematic constraint of fibre-inextensibility. Not only are they arbitrary in the sense of being independent of the magnitudes of the strain, but they can also be singular, in which case the fibre carries a non-zero resultant force. Then integration of (2.7) shows that $\mathbf{F}$ is discontinuous across such a fibre, with the jump in $\mathbf{F}$ being parallel to the fibre and representing the tensile force at each point of that fibre.

S denotes a shearing stress component. When the sheet is unstiffened

$$S = O. \tag{2.8}$$

If the sheet is elastically stiffened, we adopt Pipkin's constitutive equation

$$S = G\mathbf{a} \cdot \mathbf{b}/|J|, \tag{2.9}$$

which is equivalent to the strain energy density W per unit initial area being given by

$$W = G(1 - |J|), \tag{2.10}$$

where G is the shear modulus.

Boundary values of $\mathbf{F}$ are determined by simply integrating the prescribed tractions $\mathbf{R}(s_0)$ with respect to arc length s_0 around the undeformed boundary Γ_0. In an obvious notation we denote *typical* values of $\mathbf{F}$ along the directed segment $AB, BC, \ldots$ of Γ_0 by $\mathbf{F}_{AB}, \mathbf{F}_{BC}, \ldots$, whilst $\mathbf{F}_A, \mathbf{F}_B, \ldots$ indicate the *specific* values of $\mathbf{F}$ at $A, B, \ldots$ For convenience we let $s_0 = O$ at the crack tip O and take $\mathbf{F} = \mathbf{O}$ there. Then $\mathbf{F}_A = \mathbf{O}$ and $\mathbf{F}_E - \mathbf{F}_C$, for example, is the resultant force acting on the boundary segment CDE with

$$\mathbf{F}_E - \mathbf{F}_C = \int_{s_C}^{s_E} \mathbf{R}(s_0)\,\mathrm{d}s_0.$$

The convenience of dead loading is that these boundary values of $\mathbf{F}$ are the boundary values of $\mathbf{F}$ in the deformed configuration also.

As a final piece of notation, we let $\Delta\mathbf{f}(X)$ denote the difference between the *boundary* values of a vector function $\mathbf{f}$ at the two ends of the $\mathbf{b}$-fibre X, and similarly write $\Delta\mathbf{f}(Y)$ for the $\mathbf{a}$-fibre Y. Thus

$$\Delta\mathbf{F}(Y = O+) = \mathbf{F}_C - \mathbf{F}_E.$$

For unstiffened networks, Rivlin[32] has shown that the $\mathbf{a}$ and $\mathbf{b}$ fields through the sheet are simply related to the boundary values of $\mathbf{F}$ through

$$K\mathbf{a} = \Delta\mathbf{F}(X), \quad L\mathbf{b} = -\Delta\mathbf{F}(Y), \tag{2.11}$$

with the unit magnitudes of $\mathbf{a}$ and $\mathbf{b}$ implying that

$$K = |\Delta\mathbf{F}(X)|, \quad L = |\Delta\mathbf{F}(Y)|. \tag{2.2}$$

The generalisation of these relations to stiffened sheets in the present case of no folds is (from [31])

$$K'\mathbf{a} = \Delta\mathbf{F}(X) - G\mathbf{k}\times\Delta\mathbf{x}(X), \quad L'\mathbf{b} = -\Delta\mathbf{F}(Y) + G\mathbf{k}\times\Delta\mathbf{x}(Y), \tag{2.13}$$

where K' and L' are equal in magnitude to the respective right-hand sides. From (2.2) we see that $\Delta\mathbf{x}(X)$ and $\Delta\mathbf{x}(Y)$ are integrals of $\mathbf{b}$ and $\mathbf{a}$ over the length of the relevant fibres. Hence (2.13) do not now give explicit expressions for $\mathbf{a}$ and $\mathbf{b}$ in terms of the boundary tractions, but are in fact a pair of coupled non-linear integral equations for $\mathbf{a}$ and $\mathbf{b}$. Their solution has been considered by Pipkin[34, 35] for G both small and large compared with the magnitude of the boundary loads.

Finally we quote the result[31] given by integrating (2.7) along the relevant fibre-directions:

$$\mathbf{F} = J^{-1}(X, Y)\{N(Y)\mathbf{a}(X) + M(X)\mathbf{b}(Y)\} + G\mathbf{k}\times\mathbf{x}(X, Y), \tag{2.14}$$

where $M(X)$ and $N(Y)$ are arbitrary. For unstiffened sheets, M and N may be determined directly from the traction boundary conditions, with

$$M(X) = \mathbf{B}\cdot\mathbf{F} = -\mathbf{a}\cdot\mathbf{k}\times\mathbf{F}, \quad N(Y) = \mathbf{A}\cdot\mathbf{F} = \mathbf{b}\cdot\mathbf{k}\times\mathbf{F}, \tag{2.15}$$

where the convenient values to take for $\mathbf{F}$ are those at the relevant points on the boundary. The equivalent results for elastically stiffened sheets are

$$M(X) = -\mathbf{a}\cdot\mathbf{k}\times(\mathbf{F} - G\mathbf{k}\times\mathbf{x}), \quad N(Y) = \mathbf{b}\cdot\mathbf{k}\times(\mathbf{F} - G\mathbf{k}\times\mathbf{x}), \tag{2.16}$$

and we note that these are identical with the "unstiffened" result (2.15) provided we replace $\mathbf{F}$ with $\mathbf{F} - G\mathbf{k}\times\mathbf{x}$ in (2.15).

3. SINGULAR STRESSES AT THE CRACK TIP

The presence of singular fibres is associated either with the deformation of boundary fibres or with the application of concentrated loads on the boundary or with fibres across which the other fibre-direction is discontinuous. This last condition applies to a crack tip where it is evident that in general both $\mathbf{a}$ and $\mathbf{b}$ directions are discontinuous. The two tip fibres EC and BD are then singular, each carrying finite loads. Moreover, unlike when the deformations are infinitesimal, these loads can themselves be discontinuous, with their magnitudes P_-, P_0, Q_- and Q_0 (refer Fig. 1b) determined either directly from simple statical equilibrium or by using (2.14) together with (2.15) or (2.16).

For example, consider the region $ABCO$ shown in Fig. 2 as a free body. Assuming the sheet to be a pure network ($G = O$) we find that equilibrium of $ABCO$ requires

$$\mathbf{F}_C - R_b\mathbf{b}_0 - P_-\mathbf{a}_- = \mathbf{0}, \tag{3.1}$$

where $\mathbf{a}_-$ and $\mathbf{b}_0$ now represent their values at the tip O, and R_b is the resultant of the tensions

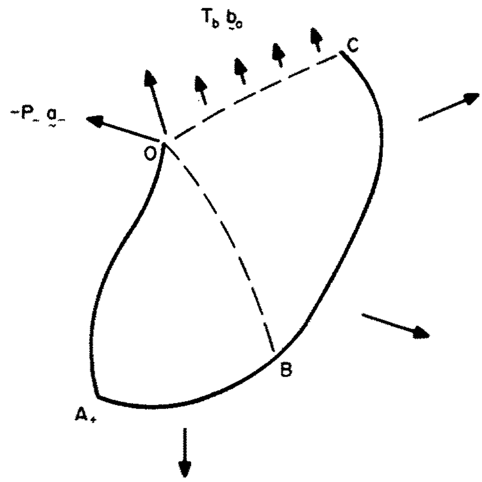

Fig. 2. Forces acting on the free body COA_+.

in the **b**-fibres acting on the tip fibre OC of length L_+:

$$R_b = \int_O^{L_+} T_b(X, O+)\,\mathrm{d}X. \tag{3.2}$$

Taking the scalar product of $\mathbf{A}_0$ with eqn (3.1) then immediately gives

$$P_- = \frac{\mathbf{F}_C \times \mathbf{b}_0 \cdot \mathbf{k}}{\mathbf{a}_- \times \mathbf{b}_0 \cdot \mathbf{k}}. \tag{3.3}$$

Alternatively, eqns (2.15) and (2.4) give

$$M(O-) = O, \quad N(O+) = \mathbf{b}_0 \cdot \mathbf{k} \times \mathbf{F}_C, \quad J(O-, O+) = \mathbf{a}_- \times \mathbf{b}_0 \cdot \mathbf{k},$$

so that

$$\mathbf{F}(O-, O+) = \frac{\mathbf{b}_0 \cdot \mathbf{k} \times \mathbf{F}_C}{\mathbf{a}_- \times \mathbf{b}_0 \cdot \mathbf{k}}\mathbf{a}_-.$$

Since the region OMA is a dead region[30], then $\mathbf{F}$ is zero there, so the discontinuity in $\mathbf{F}$ across the fibre OE at O is

$$-P_-\mathbf{a}_- = \mathbf{O} - \mathbf{F}(O-, O+)$$

yielding the required result (3.3).

Similarly, consideration of regions $ODEA$, OEA and OAB as free bodies shows that the remaining tip-forces are given by

$$Q_- = -\frac{\mathbf{F}_D \times \mathbf{a}_0 \cdot \mathbf{k}}{\mathbf{b}_- \times \mathbf{a}_0 \cdot \mathbf{k}},$$

$$P_0 = \frac{1}{J_0}\left(\mathbf{F}_E \times \mathbf{b}_0 \cdot \mathbf{k} - \frac{\mathbf{b}_- \times \mathbf{b}_0 \cdot \mathbf{k}}{\mathbf{b}_- \times \mathbf{a}_0 \cdot \mathbf{k}}\mathbf{F}_D \times \mathbf{a}_0 \cdot \mathbf{k}\right)$$

$$Q_0 = \frac{1}{J_0}\left(\mathbf{F}_B \times \mathbf{a}_0 \cdot \mathbf{k} - \frac{\mathbf{a}_- \times \mathbf{a}_0 \cdot \mathbf{k}}{\mathbf{a}_- \times \mathbf{b}_0 \cdot \mathbf{k}}\mathbf{F}_C \times \mathbf{b}_0 \cdot \mathbf{k}\right)$$

with

$$J_0 = \mathbf{a}_0 \times \mathbf{b}_0 \cdot \mathbf{k}.$$

In all these relations, the quantities $\mathbf{a}_0$, $\mathbf{b}_0$, $\mathbf{a}_-$, $\mathbf{a}_+$, $\mathbf{b}_-$ and $\mathbf{b}_+$ are evaluated at $X = O$ and $Y = O$. Conditions (2.11) and (2.12) show that in terms of the boundary tractions they can be determined as

$$\mathbf{a}_0 = \frac{\mathbf{F}_D - \mathbf{F}_B}{|\mathbf{F}_D - \mathbf{F}_B|}, \quad \mathbf{b}_0 = \frac{\mathbf{F}_E - \mathbf{F}_C}{|\mathbf{F}_E - \mathbf{F}_C|}, \quad \mathbf{a}_- = \frac{\mathbf{F}_D}{|\mathbf{F}_D|},$$
$$\mathbf{b}_- = -\frac{\mathbf{F}_C}{|\mathbf{F}_C|}, \quad \mathbf{a}_+ = -\frac{\mathbf{F}_B}{|\mathbf{F}_B|}, \quad \mathbf{b}_+ = \frac{\mathbf{F}_E}{|\mathbf{F}_E|}. \tag{3.4}$$

Then substitution and some elementary manipulations give

$$P_0 = \frac{\mathbf{F}_E \times \mathbf{F}_C \cdot \mathbf{k}}{(\mathbf{F}_D - \mathbf{F}_B) \times \mathbf{F}_C \cdot \mathbf{k}} \frac{(\mathbf{F}_D - \mathbf{F}_C) \times (\mathbf{F}_D - \mathbf{F}_B) \cdot \mathbf{k}}{(\mathbf{F}_E - \mathbf{F}_C) \times (\mathbf{F}_D - \mathbf{F}_B) \cdot \mathbf{k}} |\mathbf{F}_D - \mathbf{F}_B|$$

$$Q_0 = -\frac{\mathbf{F}_B \times \mathbf{F}_D \cdot \mathbf{k}}{(\mathbf{F}_E - \mathbf{F}_C) \times \mathbf{F}_D \cdot \mathbf{k}} \frac{(\mathbf{F}_D - \mathbf{F}_C) \times (\mathbf{F}_E - \mathbf{F}_C) \cdot \mathbf{k}}{(\mathbf{F}_D - \mathbf{F}_B) \times (\mathbf{F}_E - \mathbf{F}_C) \cdot \mathbf{k}} |\mathbf{F}_E - \mathbf{F}_C|$$

$$P_- = \frac{\mathbf{F}_C \times \mathbf{F}_E \cdot \mathbf{k}}{\mathbf{F}_D \times (\mathbf{F}_E - \mathbf{F}_C) \cdot \mathbf{k}} |\mathbf{F}_D|$$

$$Q_- = \frac{\mathbf{F}_D \times \mathbf{F}_B \cdot \mathbf{k}}{\mathbf{F}_C \times (\mathbf{F}_B - \mathbf{F}_D) \cdot \mathbf{k}} |\mathbf{F}_C|. \tag{3.5}$$

Thus the loads carried by the fibres at the crack tip may be expressed directly in terms of the given boundary tractions, for any geometry and boundary conditions.

These results (3.5) also indicate that the singular tip stresses are independent of the *shape* of the tear. However, this observation is subject to the restriction that the tear meets the tip fibres BD and CE only at the tip and nowhere else. Otherwise singular stresses would occur not only along the tip fibres but also along the fibre originally tangential to the tear. Thus, in the case illustrated in Fig. 3, not only are C_1E_1 and OD_1 singular but so too are B_2D_2 and C_2E_2; furthermore, now OB_1 is non-singular. The relevant analysis is still straightforward but more complicated and is omitted for convenience.

4. ENERGY RELEASE RATE

To determine the amount of energy dissipated per unit length of crack advance, we consider the tear extended by an infinitesimal length ϵ with components ϵ_a and ϵ_b in the two fibre directions. Then the new crack tip is at the particle $\mathbf{X} = (\epsilon_a, \epsilon_b)$ with $\epsilon^2 = \epsilon_a^2 + \epsilon_b^2$.

With zero stiffening of the sheet, the total strain energy is zero; hence the total energy of the

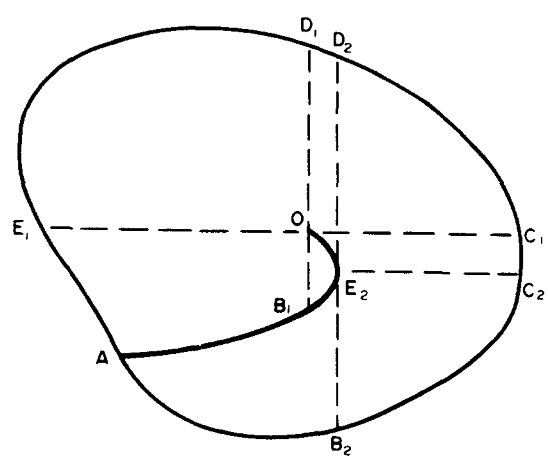

Fig. 3. Tear bending back on itself.

system is the energy of the dead loads on the boundary:

$$E = -\oint_{\Gamma} \mathbf{x} \cdot d\mathbf{F}, \tag{4.1}$$

where the line integral is taken counterclockwise around the perimeter Γ of the sheet. In terms of the initial configuration, we write

$$E = -\oint_{\Gamma_0} \mathbf{x}(\mathbf{X}) \cdot d\mathbf{F}(\mathbf{X}). \tag{4.2}$$

As the tear extends, Γ changes and energy is released.

We consider first the case when the tear extends by ϵ_a in the positive direction OC along the $\mathbf{a}$-fibre passing through the crack tip; then the new crack tip O' is at the point (ϵ_a, O). When ϵ_a is positive, the orientation of the $\mathbf{a}$-fibres as they pass through all the points in the infinitesimal band $O \leqslant X \leqslant \epsilon_a$ changes by $\delta\mathbf{a}$ where

$$\delta\mathbf{a} = \begin{cases} \mathbf{a}_+ - \mathbf{a}_0 & Y < O \\ \mathbf{a}_- - \mathbf{a}_0 & Y > O. \end{cases} \tag{4.3}$$

Hence all the points on the perimeter Γ move by $\delta\mathbf{x}$ where

$$\delta\mathbf{x} = \begin{cases} (\mathbf{a}_0 - \mathbf{a}_+)\epsilon_a & \text{on } OAB \\ \mathbf{O} & \text{on } BCD \\ (\mathbf{a}_0 - \mathbf{a}_-)\epsilon_a & \text{on } DEAO \end{cases} \tag{4.4}$$

and E changes by δE with

$$\delta E \sim -\oint_{\Gamma_0} \delta\mathbf{x}(\mathbf{X}) \cdot d\mathbf{F}(\mathbf{X}).$$

Thus the energy release rate is

$$\begin{aligned} \mathscr{G}_a &= -\lim_{\epsilon_a \to O} (\delta E/\epsilon_a) \\ &= (\mathbf{a}_0 - \mathbf{a}_+) \cdot \mathbf{F}_B + (\mathbf{a}_0 - \mathbf{a}_-) \cdot (-\mathbf{F}_D) \\ &= \mathbf{a}_- \cdot \mathbf{F}_D - \mathbf{a}_+ \cdot \mathbf{F}_B - \mathbf{a}_0 \cdot (\mathbf{F}_D - \mathbf{F}_B) \\ &= |\mathbf{F}_D| + |\mathbf{F}_B| - |\mathbf{F}_D - \mathbf{F}_B|. \end{aligned} \tag{4.5}$$

In contrast, when ϵ_a is negative (so that the tear bends back on itself by extending in the opposite direction along OD) the $\mathbf{a}$-direction would *not* change in the band $\epsilon_a < X < O$ and therefore $\mathscr{G}_a$ would then be zero.

Similarly the energy release rate for crack extension in the positive $\mathbf{b}$-fibre direction OD is

$$\mathscr{G}_b = |\mathbf{F}_E| + |\mathbf{F}_C| - |\mathbf{F}_E - \mathbf{F}_C| \tag{4.6}$$

and zero if the crack were to extend along OB. In both cases $\mathscr{G}_a$ and $\mathscr{G}_b$ are obviously non-negative.

For a general direction of crack advance, the points on Γ move by $\delta\mathbf{x}$ where

$$\delta\mathbf{x} = \begin{cases} (\mathbf{a}_0 - \mathbf{a}_+)\epsilon_a^{\,+} + (\mathbf{b}_0 - \mathbf{b}_-)\epsilon_b^{\,+} & \text{on } OAB \\ (\mathbf{b}_0 - \mathbf{b}_-)\epsilon_b^{\,+} & \text{on } BC \\ \mathbf{O} & \text{on } CD \\ (\mathbf{a}_0 - \mathbf{a}_-)\epsilon_a^{\,+} & \text{on } DE \\ (\mathbf{a}_0 - \mathbf{a}_-)\epsilon_a^{\,+} + (\mathbf{b}_0 - \mathbf{b}_+)\epsilon_b^{\,+} & \text{on } EA \end{cases} \tag{4.7}$$

with ϵ_a^+ denoting the larger of ϵ_a or zero, and similarly for ϵ_b^+. Then the rate of energy release is

$$\mathscr{G} = +\lim_{\epsilon \to O} \oint_{\Gamma_0} \delta \mathbf{x} \cdot \mathrm{d}\mathbf{F}$$

$$= \mathscr{G}_a(\cos\theta)^+ + \mathscr{G}_b(\sin\theta)^+. \tag{4.8}$$

Here the angle θ is defined through

$$\cos\theta = \lim_{\epsilon \to O}\left(\frac{\epsilon_a}{\epsilon}\right), \qquad \sin\theta = \lim_{\epsilon \to O}\left(\frac{\epsilon_b}{\epsilon}\right); \tag{4.9}$$

we note that it is *not* the angle of actual advance in the deformed configuration.

5. ELASTIC STIFFENING

The previous two sections treat the case of a reinforced sheet with negligible strength apart from its two families of reinforcing fibres. The equivalent results for a sheet with non-zero elastic stiffening are much more complicated though taking a comparatively simple form if Pipkin's strain energy function (2.10) is assumed. Then (3.3), for example, is replaced by

$$P_- = \frac{\mathbf{b}_0 \cdot \mathbf{k} \times (\mathbf{F}_C - G\mathbf{k} \times \mathbf{x}_C)}{\mathbf{b}_0 \cdot \mathbf{k} \times \mathbf{a}_-}, \tag{5.1}$$

and the relevant expression in (3.5) is

$$P_- = \frac{\mathbf{F}_C \times \mathbf{F}_E \cdot \mathbf{k} + G(\mathbf{F}_E \cdot \mathbf{x}_C - \mathbf{F}_C \cdot \mathbf{x}_E) + G^2 \mathbf{x}_C \times \mathbf{x}_E \cdot \mathbf{k}}{\mathbf{F}_D \times (\mathbf{F}_E - \mathbf{F}_C) \cdot \mathbf{k} + G\{(\mathbf{F}_E - \mathbf{F}_C) \cdot \mathbf{x}_D - \mathbf{F}_D \cdot (\mathbf{x}_E - \mathbf{x}_C)\} + G^2 \mathbf{x}_D \times (\mathbf{x}_E - \mathbf{x}_C) \cdot \mathbf{k}} K'_- \tag{5.2}$$

where (2.13) shows that

$$K'_- = |\mathbf{F}_D - G\mathbf{k} \times \mathbf{x}_D|. \tag{5.3}$$

These expressions are given by simple substitution of $\mathbf{F} - G\mathbf{k} \times \mathbf{x}$ for $\mathbf{F}$ wherever it appears in the corresponding quantity for pure networks. Equivalent expressions can be obtained for Q_-, P_0 and Q_0. They show that the loads carried at the crack tip can still be expressed directly in terms of boundary values, but not now only in terms of the given tractions.

The generalisation of expression (4.5), giving the energy release rate associated with crack propagation in the $\mathbf{a}$-direction, is

$$\mathscr{G}_a = |\mathbf{F}_D - G\mathbf{k} \times \mathbf{x}_D| + |\mathbf{F}_B - G\mathbf{k} \times \mathbf{x}_B| - |\mathbf{F}_D - \mathbf{F}_B - G\mathbf{k} \times (\mathbf{x}_D - \mathbf{x}_B)|. \tag{5.4}$$

This result was brought to my attention by A. C. Pipkin, and can be deduced from an extension of the analysis leading to (4.5). The total energy of the system is now expressible as

$$E = \int_{\mathscr{R}_0} W \,\mathrm{d}A - \oint_{\Gamma} \mathbf{x} \cdot \mathrm{d}\mathbf{F}$$

$$= G(A_0 - A) - \oint_{\Gamma} \mathbf{x} \cdot \mathrm{d}\mathbf{F},$$

where A_0 and A are the areas of the sheet in its undeformed and deformed states respectively. Applying Green's theorem to the plane area A, in this expression, we obtain

$$E = GA_0 - \tfrac{1}{2}G \oint_{\Gamma} \mathbf{k} \cdot \mathbf{x} \times \mathrm{d}\mathbf{x} - \oint_{\Gamma} \mathbf{x} \cdot \mathrm{d}\mathbf{F}$$

to replace (4.5). As the tear extends by ϵ_a in the a-direction, the energy changes by

$$\begin{aligned}\delta E &= -\oint_{\Gamma_0} \{\tfrac{1}{2}G\mathbf{k}\cdot\delta\mathbf{x}(\mathbf{X})\times d\mathbf{x}(\mathbf{X})+\tfrac{1}{2}G\mathbf{k}\cdot\mathbf{x}(\mathbf{X})\times\delta(d\mathbf{x})+\delta\mathbf{x}(\mathbf{X})\cdot d\mathbf{F}\}\\ &= -\tfrac{1}{2}G\oint_{\Gamma_0} d(\mathbf{k}\cdot\mathbf{x}\times\delta\mathbf{x})+\oint_{\Gamma_0}\{G\mathbf{k}\cdot d\mathbf{x}\times\delta\mathbf{x}-\delta\mathbf{x}\cdot d\mathbf{F}\}\\ &= \oint_{\Gamma_0}\delta\mathbf{x}\cdot d(G\mathbf{k}\times\mathbf{x}-\mathbf{F}),\end{aligned}$$

where $\delta\mathbf{x}$ is defined by (4.4) and (4.3) as before. Hence $\mathbf{F}-G\mathbf{k}\times\mathbf{x}$ again replaces $\mathbf{F}$ in the previous analysis, giving the result (5.4).

A similar expression holds for $\mathscr{G}_b$, the energy release rate for the tear's extending in the b-direction. The result for a general direction of tear extension is then immediately given by (4.8). Again we note that all these expressions are still given in terms of boundary values.

6. STIFF SHEETS

Comparison of (5.4) with expressions such as (5.2) and (5.3) shows that there are no simple relations between the energy release rates $\mathscr{G}_a$, $\mathscr{G}_b$, $\mathscr{G}$ and the tip forces P_0, P_-, Q_0 and Q_-. This is in contrast with the situation pertaining in linear elasticity, in which the energy release rates can be simply expressed[9, 36] in terms of the stress intensity factors at the crack tip. It is in particular contrast with the predictions for linearly elastic sheets with inextensible cords, for which Pipkin and Rogers[28] have deduced that

$$\mathscr{G}_a = Q_0^2/GH^*,\quad \mathscr{G}_b = P_0^2/GL^*, \tag{6.1}$$

with

$$P_0 = P_-,\quad Q_0 = Q_-, \tag{6.2}$$

and where H^* is the harmonic mean of the two segments H_+ and H_- of the b-fibre passing through the crack tip, and L^* refers to the equivalent a-fibres:

$$2/H^* = 1/H_+ + 1/H_-,\quad 2/L^* = 1/L_+ + 1/L_-. \tag{6.3}$$

In this section we show that in fact there is no contradiction between these results and those obtained by a suitable linearisation of the finite elasticity results. The linear theory should be appropriate for problems in which the shear modulus G is large compared with the applied tractions, with consequentially infinitesimal displacement at the boundaries. In that case, the position vectors of B, C, D and E are approximated by

$$\mathbf{x}_B \sim -H_-\mathbf{j},\quad \mathbf{x}_C \sim L_+\mathbf{i},\quad \mathbf{x}_D \sim H_+\mathbf{j},\quad \mathbf{x}_E \sim -L_-\mathbf{i}. \tag{6.4}$$

Then

$$\mathbf{x}_D\times(\mathbf{x}_E-\mathbf{x}_C)\cdot\mathbf{k} \sim (L_+ + L_-)H_+,$$

and for sufficiently large G the denominator of (5.2) is therefore dominated by

$$G^2H_+(L_+ + L_-),$$

the numerator by

$$G(\mathbf{F}_E\cdot\mathbf{x}_C - \mathbf{F}_C\cdot\mathbf{x}_E) \sim G(L_+F_{E1} + L_-F_{C1})$$

and

$$K_-' \sim |G\mathbf{k}\times\mathbf{x}_D| \sim GH_+.$$

Here F_{E1} and F_{C1} denote the X-components of $\mathbf{F}_E$ and $\mathbf{F}_C$; the Y-components will be written as F_{E2}, F_{C2}, etc.

Hence we obtain

$$P_- \sim \frac{L_+F_{E1} + L_-F_{C1}}{L_+ + L_-}.$$

A similar analysis shows that P_0 has the same asymptotic limit, and also that $Q_0 \sim Q_-$, thus confirming (6.2).

To obtain the asymptotic values of $\mathscr{G}_a$ and $\mathscr{G}_b$ for large G we note that, for example,

$$\begin{aligned}|\mathbf{F}_E - G\mathbf{k}\times\mathbf{x}_E| \sim |\mathbf{F}_E + GL_-\mathbf{j}| &= \{F_{E1}^2 + (F_{E2} + GL_-)^2\}^{\frac{1}{2}} \\ &= GL_-\left\{1 + \frac{F_{E2}}{GL_-} + \frac{F_{E1}^2}{2G^2L_-^2} + O(G^{-3})\right\}\end{aligned}$$

so that

$$\begin{aligned}\mathscr{G}_b &= |\mathbf{F}_E - G\mathbf{k}\times\mathbf{x}_E| + |\mathbf{F}_C - G\mathbf{k}\times\mathbf{x}_C| - |\mathbf{F}_E - \mathbf{F}_C - G\mathbf{k}\times(\mathbf{x}_E - \mathbf{x}_C)| \\ &\sim GL_- + GL_+ - G(L_+ + L_-) + F_{E2} + F_{C2} - (F_{E2} + F_{C2}) + \frac{1}{2G}\left\{\frac{F_{E1}^2}{L_-} + \frac{F_{E1}^2}{L_+} - \frac{(F_{E1} - F_{C1})^2}{L_+ + L_-}\right\} \\ &\sim P_0^2/GL^*\end{aligned}$$

as required. A similar analysis confirms the remainder of (6.1).

7. FRACTURE CRITERIA

For finite deformations of highly anisotropic sheets, we propose three fracture criteria which are analogous to the stress intensity and energy release rate criteria of linear elasticity.

The simplest and most obvious criterion to specify is that based on tensile failure of the fibres. This states that a crack will propagate when a *tensile fibre-force* reaches a critical value; this critical value need not be the same for each family of fibres. Thus we say that the tear will extend instantaneously along the **b**-direction, breaking **a**-fibres, when either P_0 or P_- reaches a critical value P_{cr}, and that it will extend in the **a**-direction when Q_0 or Q_- reaches a possibly different critical value Q_{cr}. With this hypothesis the tear will continue in a stair-step shape (referred to the original configuration), though possibly the "steps" will be sufficiently small to give a relatively smooth trajectory.

Since the singular stresses occur along the entire length of both tip-fibres, a tensile fibre-force criterion would predict that a relative weakness in either of them could lead to a fracture elsewhere than at the end of the tear; this would result in another crack developing in the sheet. For this possibility we require the variations of the fibre-forces along OB, OC, OD and OE, determined as follows. For illustration we consider OC. Equation (2.15) gives

$$M(X) = \mathbf{k}\cdot\mathbf{a}_0(X)\times\mathbf{F}_{CD}(X), \qquad N(O) = \begin{cases} -\mathbf{k}\cdot\mathbf{b}_0\times\mathbf{F}_C, & Y = O+, \\ -\mathbf{k}\cdot\mathbf{b}_-\times\mathbf{F}_C = O, & Y = O-, \end{cases} \tag{7.1}$$

for an unstiffened network, with a_0, b_0 and b_- determined from (2.11) and (2.12). Then, using (2.14), the fibre force along OC is given by

$$\begin{aligned}\mathbf{F}(X, O+) - \mathbf{F}(X, O-) = \{&(\mathbf{F}_C\times\mathbf{b}_0\cdot\mathbf{k})\mathbf{a}_0(X) - (\mathbf{F}_{CD}\times\mathbf{a}_0(X)\cdot\mathbf{k})\mathbf{b}_0\}/J_0 \\ &+ (\mathbf{F}_{CD}\times\mathbf{a}_0(X)\cdot\mathbf{k})\mathbf{b}_-/J_-,\end{aligned} \tag{7.2}$$

where

$$J_0 = \mathbf{a}_0(X)\times\mathbf{b}_0\cdot\mathbf{k}, \quad J_- = \mathbf{a}_0(X)\times\mathbf{b}_-\cdot\mathbf{k}.$$

Equation (7.2) reduces to

$$\mathbf{F}(X, O+) - \mathbf{F}(X, O-) = P_0(X)\mathbf{a}_0(X)$$

where the magnitude of the fibre force in OC is

$$P_0(X) = \frac{\mathbf{F}_E \times \mathbf{F}_C \cdot \mathbf{k}}{\mathbf{a}_0(X) \times \mathbf{F}_C \cdot \mathbf{k}} \frac{\mathbf{a}_0(X) \times \{\mathbf{F}_{CD}(X) - \mathbf{F}_C\} \cdot \mathbf{k}}{\mathbf{a}_0(X) \times (\mathbf{F}_E - \mathbf{F}_C) \cdot \mathbf{k}} \tag{7.3}$$

and

$$\mathbf{a}_0(X) = \frac{\Delta \mathbf{F}(X)}{|\Delta \mathbf{F}(X)|} = \frac{\mathbf{F}_{CD}(X) - \mathbf{F}_{BC}(X)}{|\mathbf{F}_{CD}(X) - \mathbf{F}_{BC}(x)|}. \tag{7.4}$$

Similar expressions may be found for the fibre forces in OB, OD and OE (and we also note that the results of (3.5) are confirmed when we substitute $X = O$ and $Y = O$ into these expressions). The generalisation to elastically stiffened sheets is again given by simply replacing $\mathbf{F}$ by $\mathbf{F} - G\mathbf{k} \times \mathbf{x}$.

Although maximum stresses are usually assumed to occur at the crack tip, it is not apparent that this is always the case for networks. Thus it is necessary to investigate the distribution of singular stress elsewhere even when the sheet has uniform fracture strength.

An alternative criterion for fracture is that relating to the *energy release rate* $\mathscr{G}$. As in isotropic elasticity this states that the crack extends when $\mathscr{G}$ attains a critical value $\mathscr{G}_{cr}$. Equation (4.8) shows that the maximum value of $\mathscr{G}$ is $\mathscr{G}_{max}$ where

$$\mathscr{G}^2_{max} = \mathscr{G}_a^2 + \mathscr{G}_b^2 \tag{7.5}$$

and is attained when θ is given by

$$\theta_{max} = \cos^{-1}(\mathscr{G}_a/\mathscr{G}_{max}) = \sin^{-1}(\mathscr{G}_b/\mathscr{G}_{max}). \tag{7.6}$$

Thus according to this criterion the crack extends when

$$\mathscr{G}_{max} = \mathscr{G}_{cr}, \tag{7.7}$$

and in the direction determined from

$$\theta = \tan^{-1}(\mathscr{G}_b/\mathscr{G}_a).$$

Then with this criterion the tear will continue in a smooth trajectory (referred to the original configuration), and its shape would be found by integrating the differential equation

$$\mathrm{d}Y/\mathrm{d}X = \mathscr{G}_b(X, Y)/\mathscr{G}_a(X, Y). \tag{7.8}$$

The third criterion for fracture is based on the observation that, for linearly elastic behaviour, the quantity which is directly related to the stress intensity factor[12, 27] in crack extension in the $\mathbf{a}$-direction is $P_0/\sqrt{L^*}$, not P_0, and $Q_0/\sqrt{H^*}$ not Q_0 for extension in the $\mathbf{b}$-direction. Pipkin[37] has therefore suggested that for linearly elastic materials, when the critical fibre force criterion seems to be intuitively reasonable, the actual criterion to use might be the *critical stress* criterion, namely that fracture occurs when

$$Q_0/\sqrt{H^*} = K_a \quad \text{or} \quad P_0/\sqrt{L^*} = K_b, \tag{7.9}$$

where K_a and K_b are stress intensity factors associated with crack advance in the $\mathbf{a}$- and $\mathbf{b}$-directions respectively.

For large deformations, the relations (7.9) are difficult to generalise—should we use P_- for

P_0, for example, and what are the appropriate values for L^* or H^*? However, we note that (6.1) shows that (7.9) could be rewritten as

$$\mathcal{G}_a = \bar{K}_a^2 \quad \text{or} \quad \mathcal{G}_b = \bar{K}_b^2, \tag{7.10}$$

where $\bar{K}_a = K_a/\sqrt{G}$ and $\bar{K}_b = K_b/\sqrt{G}$. This suggests that a possible, simple generalisation of the critical stress criterion which would be appropriate for finite deformations is (7.10) itself, with $\bar{K}_a$ and $\bar{K}_b$ simply being two material parameters. Such a criterion, though expressed in terms of critical energy release rates, would produce the stair-step shape of tear (referred to the original configuration) similar to that predicted by the critical fibre-force criterion.

8. TEARING OF A RECTANGULAR SHEET

As a simple example we consider the case of uniform biaxial tensile loading $T_1\mathbf{i}$ and $T_2\mathbf{j}$, per unit length, on the edges of the initially rectangular sheet $-L_- \leq X \leq L_+$, $-H_- \leq Y \leq H_+$ as shown in Fig. 4. The sheet is unstiffened and the tear extends from $(-l, -H_-)$ to (O, O), with $l \geq O$, and lies within the bounding rectangle $-l \leq X \leq O$, $-H_- \leq Y \leq O$.

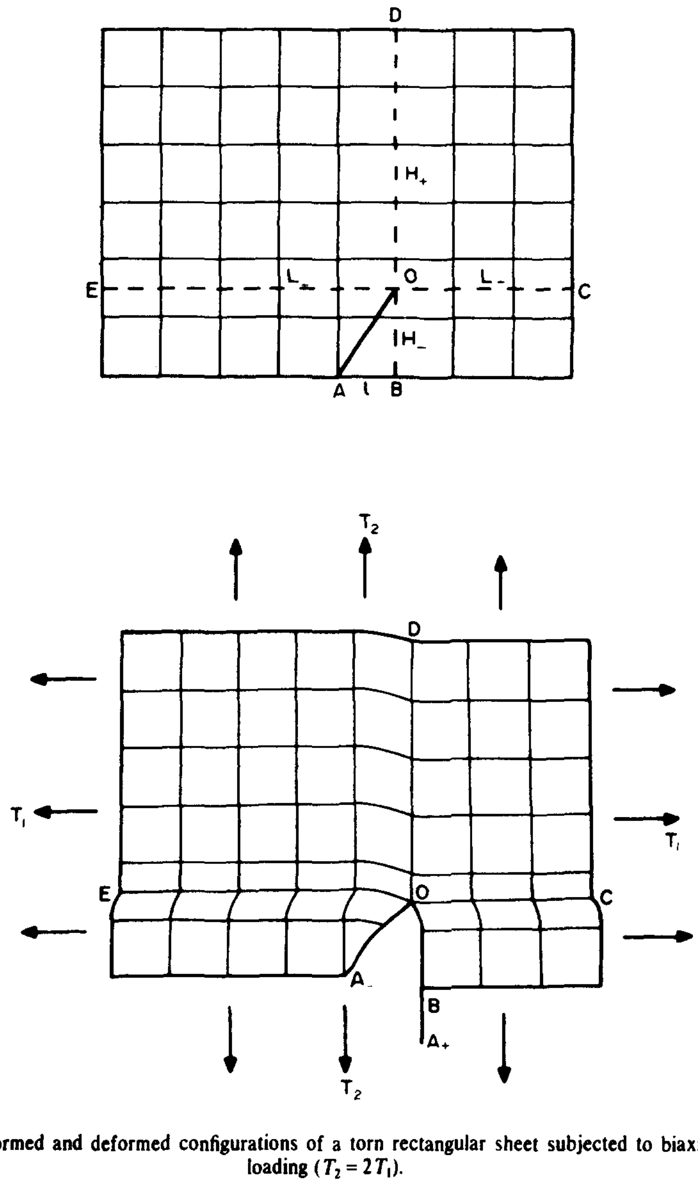

Fig. 4. Undeformed and deformed configurations of a torn rectangular sheet subjected to biaxial tensile loading ($T_2 = 2T_1$).

With $\mathbf{F} = \mathbf{O}$ at the crack tip O, integration of the boundary tractions gives

$$\mathbf{F}_B = -lT_2\mathbf{j}, \quad \mathbf{F}_C = H_-T_1\mathbf{i} - (L_+ + l)\mathbf{j}, \quad \mathbf{F}_D = HT_1\mathbf{i} - lT_2\mathbf{j},$$
$$\mathbf{F}_E = H_-T_1\mathbf{i} + (L_- - l)T_2\mathbf{j}, \tag{8.1}$$

where $H = H_+ + H_-$, the width of the sheet. Substitution into (3.5) then yields

$$P_0 = L_+H_-T_1/(L_+ + l), \qquad Q_0 = lH_+T_2/H,$$
$$P_- = (H^2T_1{}^2 + l^2T_2{}^2)^{\frac{1}{2}}H_-/H, \quad Q_- = \{H_-{}^2T_1{}^2 + (L_+ + l)^2T_2{}^2\}^{\frac{1}{2}}l/(L_+ + l) \tag{8.2}$$

and (4.5) and (4.6) give

$$\mathcal{G}_a = (H^2T_1{}^2 + l^2T_2{}^2)^{\frac{1}{2}} + lT_2 - HT_1,$$
$$\mathcal{G}_b = \{H_-{}^2T_1{}^2 + (L_- - l)^2T_2{}^2\}^{\frac{1}{2}} + \{H_-{}^2T_1{}^2 + (L_+ + l)^2T_2{}^2\}^{\frac{1}{2}} - LT_2. \tag{8.3}$$

The implications for fracture are straightforward. Equations (8.2) show that P_- and Q_- are greater than P_0 and Q_0 respectively. Hence the fibre-force criterion would predict that extension of the tear would occur in the **b**-direction (by breaking the **a**-fibre) when T_1 and T_2 increase to values such that P_- attains P_{cr} first, before Q_- reaches the other critical value Q_{cr}; these conditions are equivalent to

$$H_-{}^2(H^2T_1{}^2 + l^2T_2{}^2) = H^2P_{cr}^2, \quad \text{with} \quad f(T_1/T_2) > O, \tag{8.4}$$

where

$$f\left(\frac{T_1}{T_2}\right) = \left\{\frac{Q_{cr}^2}{P_{cr}^2} - \frac{l^2}{(L_+ + l)^2}\right\}\frac{T_1{}^2}{T_2{}^2} + \left(\frac{Q_{cr}^2}{P_{cr}^2} - \frac{H^2}{H_-{}^2}\right)\frac{l^2}{H^2}. \tag{8.5}$$

Similarly fracture would occur along the **a**-direction when

$$l^2\{H_-{}^2T_1{}^2 + (L_+ + l)^2T_2{}^2\} = (L_+ + l)^2Q_{cr}^2, \quad \text{with} \quad f(T_1/T_2) < O. \tag{8.6}$$

The shape of any subsequent tearing can be controlled by suitable changing of the boundary tractions T_1 and T_2. For example, suppose that conditions (8.6) were satisfied (with fracture initiating in the **a**-direction) and that T_1 and T_2 were subsequently controlled so that T_1 remained constant and T_2 were suitably reduced in order to satisfy (8.6). Then tearing would continue to take place in the **a**-direction with P_- actually *decreasing* as l increased. If, however, T_1 and T_2 were controlled so that T_1 were increased and T_2 decreased such that (8.6) continued to be satisfied, then tearing would continue in the **a**-direction only until l were such that $f(T_1/T_2)$ became zero; subsequent tearing would then take place in the **b**-direction, with T_1 and T_2 controlled to satisfy (8.4).

If the tractions were left unaltered from their values at initiation of fracture, then catastrophic tearing would result. For example, if conditions (8.4) were satisfied, then initial fracture would result in H_- increasing; if T_1 and T_2 were not suitably changed to take this increase into account, then P_- would increase, implying further extension of the tear in the **b**-direction, and hence further increase in H_-. Though Q_- might also increase, (8.5) shows that the crack extension would not change direction, since $f(T_1/T_2)$ increases with increasing H_-. Similarly, if the conditions (8.6) were to be attained, then catastrophic tearing would be initiated along the **a**-direction. However, (8.5) and (8.4) show that, for the special case $l = O$ (a tear coinciding with the **b**-direction), this citerion predicts that fracture, if it occurs, will *always* continue in the **b**-direction. We also note that conditions (8.4)–(8.6) are all independent of L_-.

The singular stresses in the tip-fibres OB, OC, OD and OE should also be determined lest the maximum force should occur elsewhere than at the crack tip. From (7.3) and (7.4) it is easily shown that the fibre force in OC is

$$P_0(X) = (L_+ - X)H_-T_1/(L_+ + l), \quad O < X \leq L_+, \tag{8.7}$$

and in this simple example it is straightforward to determine the other forces as

$$Q_0(Y)=(H_+-Y)lT_2/H, \qquad O<Y\leqslant H_+ \tag{8.8}$$

$$P_-(X)=\begin{cases}\{H^2T_1^2+(X+l)^2T_2^2\}^{\frac{1}{2}}H_-/H, & -l\leqslant X<O\\ (L_-+X)H_-T_1/(L_--l), & -L_-\leqslant X\leqslant -l\end{cases} \tag{8.9}$$

$$Q_-(Y)=\{(L_++l)^2T_2^2+(Y+H_-)^2T_1^2\}^{\frac{1}{2}}l/(L_++l), \qquad -H_-\leqslant Y\leqslant O. \tag{8.10}$$

Clearly in each case the maximum value occurs at the crack tip so if the fracture strength parameters P_{cr} and Q_{cr} are independent of position then the fibre force criterion would show that fracture can occur only at the tip.

The alternative fracture criterion (7.7) based on a maximum energy release rate predicts extension of the tear when T_1 and T_2 increase to values such that

$$\mathcal{G}_a^2+\mathcal{G}_b^2=\mathcal{G}_{cr}^2, \tag{8.11}$$

where $\mathcal{G}_a$ and $\mathcal{G}_b$ are given by (8.3); the direction θ of extension of the tear would be $\tan^{-1}(\mathcal{G}_b/\mathcal{G}_a)$. The subsequent crack locus would then be traced by the point (X', Y') given by

$$\frac{\mathrm{d}Y'}{\mathrm{d}X'}=\frac{\{(Y'+H_-)^2+t^2(L_--l)^2\}^{\frac{1}{2}}+\{(Y'+H_-)^2+t^2(L_++l)^2\}^{\frac{1}{2}}-tL}{\{t^2(X'+l)^2+H^2\}^{\frac{1}{2}}+t(X'+l)-H}$$

with $Y'=O$ when $X'=O$ and where $t=t(X', Y')$ is given implicitly through

$$\{(t^2X'^2+H^2)^{\frac{1}{2}}+t(X'+l)-H\}^2$$
$$+[\{(Y'+H_-)^2+t^2(L_--l)^2\}^{\frac{1}{2}}+\{(Y'+H_-)^2+t^2(L_++l)^2\}^{\frac{1}{2}}-tL]^2=\mathcal{G}_{cr}^2/T_1^2.$$

The conditions for fracture predicted by the two different criteria are obviously quite different. Only in the special limit of almost uniaxial tension ($T_2/T_1\to O$) are the predictions similar, with the energy release rate criterion then also predicting tearing along the **b**-direction. The conditions become even more dissimilar in form when elastic stiffening is incorporated into the theory.

The third criterion—the generalised stress intensity criterion (7.10)—predicts the same distinctive stair-step shape as the fibre-force criterion. However, the actual shapes can be quite different. For the rectangular sheet considered above, this new criterion predicts that extension of the tear would occur in the **a**-direction when T_1 and T_2 are such that $\mathcal{G}_a$ attains $\bar{K}_a^2$ first, before $\mathcal{G}_b$ reaches the other critical value $\bar{K}_b^2$; these conditions are equivalent to

$$2lT_2(\bar{K}_a^2+HT_1)=\bar{K}_a^2(\bar{K}_a^2+2HT_1) \tag{8.12}$$

with

$$\mathcal{G}_b=\{H_-^2T_1^2+(L_--l)^2T_2^2\}^{\frac{1}{2}}+\{H_-^2T_1^2+(L_++l)^2T_2^2\}^{\frac{1}{2}}-LT_2<\bar{K}_b^2; \tag{8.13}$$

similar conditions can be found for extension in the **b**-direction. If T_1 is subsequently kept constant and T_2 suitably reduced for controlled extension, then (8.12) and (8.13) together show that $\mathcal{G}_b$ *increases* with the consequentially increasing l; hence the tearing will now continue in the **a**-direction until l becomes such that $\mathcal{G}_b$ equals $\bar{K}_b^2$. Any further tearing would then take place in the **b**-direction, since $\mathcal{G}_a$ would then decrease from its critical value $\bar{K}_a^2$. Such a change in direction is in sharp contrast with the continued tearing in the **a**-direction predicted by the fibre-force criterion.

9. DISCUSSION

The preceding sections demonstrate the advantages of the inextensible network theory in that its simplicity allows analytical solutions and predictions to be produced for otherwise insoluble boundary value problems. Nevertheless it is necessarily only an idealised model for real highly anisotropic sheets. As such its results must be interpreted with care and caution, whilst also recognising that in other contexts the inextensible-fibre theory has provided valuable and correct predictions for the behaviour of real materials.

The results emphasise the difference between the behaviour of strongly anisotropic sheets and the corresponding and better understood behaviour of isotropic materials. This difference is particularly marked in its implications for fracture.

In fracture theory the conventional procedure is to determine functions of the applied loads and geometry of a cracked body which are believed to characterise the severity of deformations around the crack tip; the most popular functions are stress intensity factors, energy release rate or, if plastic deformation is prominent, the extent of the surrounding plastic zone[36]. Fracture is then predicted when one or more of these functions attain critical values, which are treated as material parameters. This is the approach which we also have adopted in this paper (Section 7). We also take the view[37] that a fracture criterion is simply a constitutive assumption for a given material and therefore expect that, like other material properties, the criterion will take different forms for different types of materials.

Only experimental investigations with highly anisotropic sheets will determine which, or indeed whether any, of the material parameters $\mathcal{G}_{cr}$, P_{cr}, Q_{cr}, $\bar{K}_a$, $\bar{K}_b$ are appropriate for predicting quasi-static fracture of highly deformable sheets. The criterion of maximum energy release rate predicts smooth crack trajectories which do not follow the fibre-firections; these may be reasonable for brittle materials, but the common-place experience of tears in fabrics suggests that the predictions are not plausible for large deformations of highly anisotropic sheets. The two parameters $\bar{K}_a$ and $\bar{K}_b$ have the attractive property that for small elastic deformations the fracture criterion (7.10) in which they are involved does reduce to the conventional critical stress criterion. Finally we remark that whenever the highly anisotropic sheets can be viewed as unstiffened or weakly stiffened nets of inextensible cords, then the obvious criterion to use would seem to be that based on maximum tensile fibre force.

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